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The state-vector space for two-mode parabosons and charged parabose coherent states

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Abstract. The structure of the state-vector space for the two-mode parabose system is investigated and a complete set of state vectors is constructed. The basis vectors are orthonormal in order $p = 2$. In order $p = 2$, conserved-charge parabose coherent states are constructed and an explicit completeness relation is obtained.

1. Introduction

A fundamental unresolved question in physics is whether all particles of nature are necessarily either bosons or fermions. Theoretical investigations of other possibilities in local, relativistic quantum field theory show that there may exist more general particle statistics [1, 2]. There may exist three types of statistics for identical particles: the parabose and parafermi statistics for which the number of particles in an antisymmetric or a symmetric state, respectively, cannot exceed a given integer p, called the order of the parastatistics; and for two space dimensions, infinite statistics based on the braid group. Practical motivation for the study of generalized statistics includes their application to the theory of the fractional quantum Hall effect [2]. In the case of parastatistics, there is the additional motivation of the possible production of $p > 1$ paraparticles at the higher energies of new and future colliders (the Tevatron, the LHC, the NLCs, etc). At present, in spite of these practical motivations, there is very little known about specific signatures for multi-mode parabosons nor is there complete information about the structure of the state-vector space [1].

Knowledge of the structure of the state-vector space for a quantum theory is essential. For example, many successful applications of the ordinary boson and fermion descriptions in various fields of physics are based on the full knowledge about the structure of the ordinary bose and fermi state-vector spaces. During the period of early interest in parastatistics, the explicit structure of the state-vector space for a single-mode of parabosons, and of parafermions, was determined. The associated coherent states for a single parabose mode were also constructed [3]. Because there are no simple commutative or anticommutative bilinear relations between single paraparticles belonging to different degrees of freedom, knowledge about the structure of the state-vector space for the case of more than one mode of paraparticles remains very limited. One knows that the space is spanned by a state vector of the form $\mathcal{M}(a_k^\dagger, a_l^\dagger, \ldots)|0\rangle$,

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where $\mathcal{M}$ denotes an arbitrary monomial in the parabose creation operators $a_k^\dagger, a_l^\dagger, \ldots$, and $|0\rangle$ is the unique vacuum state. While this n-paraparticle state vector can be written [4] as a so-called 'state vector of the standard form', one still does not know the explicit form of a complete set of basis vectors for such systems. Consequently, parabose coherent states for more than the one-mode case have not been constructed.

In this paper, we investigate the simplest non-trivial case—the structure of the state space for the two-mode parabose system. While the fundamental parabose commutation relations are trilinear,

$$\begin{aligned} &[a_k, \{a_l^\dagger, a_m\}] = 2\delta_{kl}a_m \qquad [a_k, \{a_l^\dagger, a_m^\dagger\}] = 2\delta_{kl}a_m^\dagger + 2\delta_{km}a_l^\dagger \\ &[a_k, \{a_l, a_m\}] = 0 \qquad (k, l, m = 1, 2) \end{aligned} \tag{1}$$

it does nevertheless follow that there are some simple commutation relations between 'A' paraboson operators and the 'B' paraboson operators: letting $a \equiv a_1$, $b \equiv a_2$,

$$[a, b^2] = 0 \qquad [b, a^2] = 0 \qquad [a^\dagger, b^2] = 0 \qquad [b^\dagger, a^2] = 0 \tag{2}$$

plus the Hermitian conjugate relations.

In section 2 we use these relations to construct an explicit complete set of state vectors for the two-mode parabose system. Then for $p = 2$ order parabosons, in section 3 we show that these state vectors are also orthogonal and thereby obtain a complete, orthonormal set of basis vectors. In section 4, we use these basis vectors to construct the conserved-charge parabose coherent states and show that they satisfy an explicit completeness relation.

2. The complete set of state vectors

We assume there is a unique vacuum state $|0\rangle$ satisfying

$$a_k|0\rangle = 0 \qquad a_k a_l^\dagger|0\rangle = p\delta_{kl}|0\rangle \tag{3}$$

and consider a state with a total number N of parabosons. Without loss of generality, in this state there are n parabosons A and m parabosons B, with $0 \leqslant n, m \leqslant N$ and $n + m = N$. We denote each partition by (n, m). For a given N, the number of its partitions is $N + 1$.

Theorem. *The dimension of the subspace* (n, m) *is* $\min(n, m) + 1$.

To prove this, without loss of generality we assume that $n \geqslant m$. Counting the number of states in the subspace (n, m) is equal to counting the number of ways of arranging n particles A and m particles B in $n + m = N$ boxes. While A and B cannot be freely interchanged, by (2) two adjacent A particles can be freely interchanged with a single B particle, and vice versa. For m even, we first assign the m particles B in the last m boxes, with the other boxes occupied by the n particles A. We call this state 1, and denote it $AA\ldots ABB\ldots B$. Without separating the B particles, no new state arises. So we next separate one B particle and put it in the last box, with one A in the next box, and then put the remaining B particles in the next $m - 1$ boxes. This is a new state (state 2) denoted by $A\ldots ABB\ldots BAB$. Again, repositioning the group of $m - 1$ B particles does not produce a state different from states 1 or 2. Next we put ABA in the last three boxes, and the $m - 1$ group of B particles in the next $m - 1$ boxes, followed by the $n - 2$ A particles. This is a new state (state 3) denoted by $A\ldots AB\ldots B\ldots BABA$. Again, repositioning the group of $m - 1$ B particles does not produce a new state. The next step is to separate three B particles on the right, and insert A particles to keep these three B particles separate, while the remaining $m - 3$ B particles are kept grouped together. Proceeding as before, this again gives two new states: $A\ldots AB\ldots BABABAB$ and $A\ldots AB\ldots BABABABABA$. Continuing this process, each

time we separate two additional B particles and obtain two different states with different right endings. This procedure ends when it produces two states with all the B particles non-adjacent. Thereby, for m even, we obtain $2[\frac{m}{2}] + 1 = m + 1$ different states†. Similarly for m odd, we construct $2[\frac{m+1}{2}] = m + 1$ different states.

This theorem reflects a major difference between the paraboson system and the ordinary boson system. In the ordinary boson case, there is only one state in the subspace (n, m), i.e. $(a^\dagger)^n(b^\dagger)^m|0\rangle$, versus $m+1$ different states (if $n \geqslant m$) in the paraboson system. It also follows from this theorem, that the dimension of the state-vector space with a total of N paraboson particles A and B is $([\frac{N}{2}] + 1)([\frac{N-1}{2}] + 2)$, instead of the dimension $N + 1$ for the ordinary two-mode boson case.

In summary, for all n and m values, we can write the state vector of n parabosons A and m parabosons B as

$$|n, m; i\rangle = \frac{1}{\sqrt{N^i_{n,m}}}(a^\dagger)^{n-i+S}(b^\dagger)^{m-2[\frac{i-S}{2}]}(a^\dagger b^\dagger)^{2[\frac{i-S}{2}]}(a^\dagger)^{i-S-2[\frac{i-S}{2}]}|0\rangle \tag{4}$$

where $N^i_{n,m}$ is the normalization constant, and

$$S = \tfrac{1}{2}(1 - (-)^m) \qquad 1 \leqslant i \leqslant \min(n, m) + 1.$$

It is useful to note that the appearance of a new index i is a characteristic feature of para-systems with more than one mode. This occurs because of the intrinsic degeneracy of the many-mode parasystem; i.e. the quantum numbers n and m do not suffice to completely specify the quantum states since $ab \neq ba$. Since the proof is constructive, there is no possibility to build any other states, and so the set of state vectors

$$\{|n, m; i\rangle | n, m = 0, 1, \ldots; 1 \leqslant i \leqslant \min(n, m) + 1\}$$

is complete.

These state vectors are orthogonal between different (n, m) subspaces, but in general the Gram–Schmidt orthogonalization method or some other procedure must be used for different i and i' states in such a subspace. However, this latter step is not necessary for order $p = 2$ as we next show.

3. The order $p = 2$ case

Inspection of the structure of the state vector given in (4) shows that there are two distinct orderings of the parabosons A and B in the (n, m) subspace:

(i) type I: $(a^\dagger)^{n-2j}(b^\dagger)^{m-2j}(a^\dagger b^\dagger)^{2j}|0\rangle$, and

(ii) type II: $(a^\dagger)^{n-2j-1}(b^\dagger)^{m-2j-1}(b^\dagger a^\dagger)^{2j+1}|0\rangle$, where $j = [\frac{i-S}{2}]$. Type I(II) respectively corresponds to $i - S$ being even (odd), where S is defined after (4).

Neglecting the normalization factors $N^i_{n,m}$ and $N^{i'}_{n,m}$, when both state vectors are type I we have

$$\langle n, m; i'|n, m; i\rangle = \langle 0|(ba)^{2j'}b^{m-2j'}a^{n-2j'}(a^\dagger)^{n-2j}(b^\dagger)^{m-2j}(a^\dagger b^\dagger)^{2j}|0\rangle \tag{5}$$

where $j' = [\frac{i'-S}{2}]$. With no loss in generality we assume $j > j'$. Using the algebraic relations

$$a(a^\dagger)^{2n} = 2n(a^\dagger)^{2n-1} + (a^\dagger)^{2n}a \qquad b(b^\dagger)^{2n} = 2n(b^\dagger)^{2n-1} + (b^\dagger)^{2n}b \tag{6}$$

† For $x \geqslant 0$, square brackets $[x]$ denote the integer part of x.

and (only true for $p = 2$)

$$ba^\dagger b^\dagger = b^\dagger a^\dagger b \qquad ab^\dagger a^\dagger = a^\dagger b^\dagger a \qquad bb^\dagger a^\dagger = a^\dagger b^\dagger b + 2a^\dagger \qquad aa^\dagger b^\dagger = b^\dagger a^\dagger a + 2b^\dagger \tag{7}$$

we get†

$$\langle n, m; i' | n, m; i\rangle = \begin{cases} \dfrac{(m-2j)!!(m+2j)!!}{(4j)!!} \\ \quad \times \langle 0|(ba)^{2j'} a^{n-2j'} (a^\dagger)^{n-2j} (b)^{2(j-j')} (a^\dagger b^\dagger)^2 |0\rangle = 0 \\ \quad (m \text{ even}) \\ \dfrac{(m-1-2j)!!(m+1+2j)!!}{(4j)!!} \\ \quad \times \langle 0|(ba)^{2j'} ba^{n-2j'} (a^\dagger)^{n-2j} (b)^{2(j-j')-1} (a^\dagger b^\dagger)^2 |0\rangle = 0 \\ \quad (m \text{ odd}). \end{cases} \tag{8}$$

In (8) we used the fact that $bb^\dagger (a^\dagger b^\dagger)^n |0\rangle = 2(n+1)(a^\dagger b^\dagger)^n |0\rangle$ and $b(a^\dagger b^\dagger)^n |0\rangle = (b^\dagger a^\dagger)^n b|0\rangle = 0$.

Next we consider the case of both the state vectors being type II:

$$\langle n, m; i' | n, m; i\rangle = \langle 0|(ab)^{2j'+1} b^{m-2j'-1} a^{n-2j'-1} (a^\dagger)^{n-2j-1} (b^\dagger)^{m-2j-1} (b^\dagger a^\dagger)^{2j+1} |0\rangle. \tag{9}$$

If m is odd, this overlap (9) can be written as

$$\langle 0|(ab)^{2j'+1} b^{m-2j'-1} (b^\dagger)^{m-2j-1} a^{n-2j'-1} (a^\dagger)^{n-2j-1} (b^\dagger a^\dagger)^{2j+1} |0\rangle$$
$$= \begin{cases} \dfrac{(n-2-2j)!!(n+2+2j)!!}{(4j+2)!!} \\ \quad \times \langle 0|(ab)^{2j'+1} b^{m-2j'-1} (b^\dagger)^{m-2j-1} a^{2(j-j')} (b^\dagger a^\dagger)^{2j+1} |0\rangle = 0 \\ \quad (n \text{ even}) \\ \dfrac{(n-1-2j)!!(n+1+2j)!!}{(4j+2)!!} \\ \quad \times \langle 0|(ab)^{2j'+1} b^{m-2j'-1} (b^\dagger)^{m-2j-1} a^{2(j-j')} (b^\dagger a^\dagger)^{2j+1} |0\rangle = 0 \\ \quad (n \text{ odd}). \end{cases} \tag{10}$$

If m is even, the overlap (9) can be written as

$$\langle 0|(ab)^{2j'+1} b^{m-2j'-1} (b^\dagger)^{m-2j} a^{n-2j'-1} (a^\dagger)^{n-2j} (b^\dagger a^\dagger)^{2j} |0\rangle$$
$$= \begin{cases} \dfrac{(n-2j)!!(n+2j)!!}{(4j)!!} \\ \quad \times \langle 0|(ab)^{2j'+1} b^{m-2j'-1} (b^\dagger)^{m-2j} a^{2(j-j')-1} (b^\dagger a^\dagger)^{2j} |0\rangle = 0 \\ \quad (n \text{ even}) \\ \dfrac{(n-1-2j)!!(n+1+2j)!!}{(4j)!!} \\ \quad \times \langle 0|(ab)^{2j'+1} b^{m-2j'-1} (b^\dagger)^{m-2j} a^{2(j-j')-1} (b^\dagger a^\dagger)^{2j} |0\rangle = 0 \\ \quad (n \text{ odd}). \end{cases} \tag{11}$$

Finally, we consider the overlap of one state vector belonging to type I and another state vector belonging to type II:

$$\langle n, m; i' | n, m; i\rangle = \langle 0|(ab)^{2j'+1} b^{m-2j'-1} a^{n-2j'-1} (a^\dagger)^{n-2j} (b^\dagger)^{m-2j} (a^\dagger b^\dagger)^{2j} |0\rangle$$

† Note $(2k)!! = 2k(2k-2)(2k-4)\ldots 2$, $(2k+1)!! = (2k+1)(2k-1)(2k-3)\ldots 1$.

$$= \begin{cases} \dfrac{(m-2j)!!(m+2j)!!}{(4j)!!}\langle 0|(ab)^{2j'}a^{n-2j'}(a^\dagger)^{n-2j}b^{2(j-j')}(a^\dagger b^\dagger)^{2j}|0\rangle = 0 \\ \qquad (m \text{ even}) \\ \dfrac{(m-1-2j)!!(m+1+2j)!!}{(4j)!!} \\ \qquad \times\langle 0|(ab)^{2j'+1}a^{n-2j'-1}(a^\dagger)^{n-2j}b^{2(j-j')-1}(a^\dagger b^\dagger)^{2j}|0\rangle = 0 \\ \qquad (m \text{ odd}). \end{cases} \tag{12}$$

Thus,

$$\langle n, m; i'|n, m; i\rangle = 0 \qquad \text{for} \quad i \neq i' \tag{13}$$

which completes the proof of orthogonality for $p = 2$ for the state vectors given by (4).

The normalization constant $N^i_{n,m}$ for the state vector $|n, m; i\rangle$ easily follows from the algebraic relations (7),

$$(N^i_{n,m})^2 = 2^{n+m}\left[\frac{n+i}{2}\right]!\left[\frac{n+1-i}{2}\right]!\left[\frac{m+i}{2}\right]!\left[\frac{m+1-i}{2}\right]!. \tag{14}$$

When the annihilation operators a and b act on this set of basis vectors, one finds

$$a|n, m; i\rangle = \begin{cases} \sqrt{2\left[\dfrac{n+i}{2}\right]}|n-1, m; i\rangle & \text{if} \quad (n+i) \text{ even} \\ \sqrt{2\left[\dfrac{n+1-i}{2}\right]}|n-1, m; i\rangle & \text{if} \quad (n+i) \text{ odd} \end{cases} \tag{15}$$

and

$$b|n, m; i\rangle = \begin{cases} \sqrt{2\left[\dfrac{m+1-i}{2}\right]}|n, m-1; i+1\rangle & \text{if} \quad (n+i) \text{ even} \\ \sqrt{2\left[\dfrac{m+i}{2}\right]}|n, m-1; i-1\rangle & \text{if} \quad (n+i) \text{ odd.} \end{cases} \tag{16}$$

When we use (16) for $i = 1$, we identify $|n, m-1; i = 0\rangle$ with $|n, m-1; i = 1\rangle$ since the construction of (4) does not include $i = 0$. For instance, since $|2, 2; 1\rangle = \frac{1}{4}(a^\dagger)^2(b^\dagger)^2|0\rangle$ and $|2, 1; 1\rangle = \frac{1}{2\sqrt{2}}(a^\dagger)^2b^\dagger|0\rangle$, we have $b|2, 2; 1\rangle = \frac{1}{2}(a^\dagger)^2b^\dagger|0\rangle = \sqrt{2}|2, 1; 1\rangle$.

Similarly, when the creation operators act on the basis vectors,

$$a^\dagger|n, m; i\rangle = \begin{cases} \sqrt{2\left[\dfrac{n+2-i}{2}\right]}|n+1, m; i\rangle & \text{if} \quad (n+i) \text{ even} \\ \sqrt{2\left[\dfrac{n+1+i}{2}\right]}|n+1, m; i\rangle & \text{if} \quad (n+i) \text{ odd} \end{cases} \tag{17}$$

and

$$b^\dagger|n, m; i\rangle = \begin{cases} \sqrt{2\left[\dfrac{m+2+i}{2}\right]}|n, m+1; i+1\rangle & \text{if} \quad (n+i) \text{ even} \\ \sqrt{2\left[\dfrac{m+3-i}{2}\right]}|n, m+1; i-1\rangle & \text{if} \quad (n+i) \text{ odd.} \end{cases} \tag{18}$$

In (18), for $i = 1$ we identify $|n, m+1; i = 0\rangle$ with $|n, m+1, i = 1\rangle$.

The parabose number operators N_a and N_b for $p = 2$ order are respectively defined by

$$N_a = \tfrac{1}{2}\{a^\dagger, a\} - 1 \qquad N_b = \tfrac{1}{2}\{b^\dagger, b\} - 1. \tag{19}$$

From (15)–(18)

$$N_a|n,m;i\rangle = n|n,m;i\rangle \qquad N_b|n,m;i\rangle = m|n,m;i\rangle \tag{20}$$

so the state vectors $|n,m;i\rangle$ are common eigenvectors of N_a, N_b, and, thus, are two-mode parabose number states.

4. Conserved-charge parabose coherent states for order $p = 2$

As an application of the complete set of orthonormal state vectors for the two-mode parabose system for $p = 2$ order, we construct the associated conserved-charge parabose coherent states. In physics applications of coherent state techniques it is normally necessary to make various approximations, but it often remains important to maintain the conservation of an Abelian charge.

Using the above number operators, N_a and N_b, we define a Hermitian charge operator by

$$Q = N_a - N_b \tag{21}$$

so each of the A quanta possesses a charge '+1' and each of the B quanta a charge '-1'. Since Q does not commute with a or b, we cannot require that the coherent state be simultaneously an eigenstate of Q and the annihilation operators a and/or b. Since

$$[Q,ab] = 0 \qquad [Q,ba] = 0 \qquad [ab,ba] = 0 \tag{22}$$

we define the conserved-charge parabose coherent state $|q,z,z'\rangle$ by the requirements that

$$Q|q,z,z'\rangle = q|q,z,z'\rangle \qquad ab|q,z,z'\rangle = z|q,z,z'\rangle \qquad ba|q,z,z'\rangle = z'|q,z,z'\rangle. \tag{23}$$

Here for parabosons since $ab \neq ba$, we introduce two complex numbers z and z', unlike for ordinary bosons ($p = 1$) where only one z was needed [5].

To obtain an explicit expression for these coherent states, we consider the expansion

$$|q,z,z'\rangle = \sum_{n,m=0}^{\infty} \sum_{i=1}^{\min(n,m)+1} c^i_{n,m}|n,m;i\rangle \tag{24}$$

with the $c^i_{n,m}$ expansion coefficents to be determined. Since $|q,z,z'\rangle$ is an eigenstate of Q, for $q \geqslant 0$

$$|q,z,z'\rangle = \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} c^i_{q+m,m}|q+m,m;i\rangle. \tag{25}$$

Substituting this expression into the remaining two eigen-equations in (23) and using (15), (16), we obtain

$$c^i_{q+m,m}(z,z') = c^1_{q,0} \frac{\sqrt{[\frac{q}{2}]![\frac{q+1}{2}]!}(z)^{[\frac{m-(-)^{q+m+i}}{2}+\frac{1-(-)^q}{4}]}(z')^{[\frac{m+(-)^{q+m+i}}{2}+\frac{1+(-)^q}{4}]}}{2^m\sqrt{[\frac{m+i}{2}]![\frac{q+m+i}{2}]![\frac{m+1-i}{2}]![\frac{q+m+1-i}{2}]!}}. \tag{26}$$

Thus, the charged parabose coherent state for $q \geqslant 0$ is

$$|q,z,z'\rangle = N_q(z,z') \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} \frac{(z)^{[\frac{m-(-)^{q+m+i}}{2}+\frac{1-(-)^q}{4}]}(z')^{[\frac{m+(-)^{q+m+i}}{2}+\frac{1+(-)^q}{4}]}}{2^m\sqrt{[\frac{m+i}{2}]![\frac{q+m+i}{2}]![\frac{m+1-i}{2}]![\frac{q+m+1-i}{2}]!}}|q+m,m;i\rangle \tag{27}$$

with the normalization constant

$$\begin{aligned}(N_q)^{-2} &= \sum_{m=0}^{\infty} \sum_{i=1}^{m+1} \frac{|z|^{2[\frac{m-(-)^{q+m+i}}{2}+\frac{1-(-)^q}{4}]}|z'|^{2[\frac{m+(-)^{q+m+i}}{2}+\frac{1+(-)^q}{4}]}}{2^{2m}[\frac{m+i}{2}]![\frac{q+m+i}{2}]![\frac{m+1-i}{2}]![\frac{q+m+1-i}{2}]!} \\ &= \left(i\frac{|z|}{2}\right)^{-[\frac{q}{2}]} J_{[\frac{q}{2}]}(i|z|)\left(i\frac{|z'|}{2}\right)^{-[\frac{q+1}{2}]} J_{[\frac{q+1}{2}]}(i|z'|)\end{aligned} \tag{28}$$

where J_n is a Bessel function of order n.

When $q < 0$, the construction proceeds similarly: the charged parabose coherent state is

$$|q, z, z'\rangle = \sum_{m=0}^{\infty}\sum_{i=1}^{m+1} c^i_{m,|q|+m}|m, |q|+m; i\rangle \tag{29}$$

where

$$c^i_{m,|q|+m}(z,z') = c^1_{0,q}\frac{\sqrt{[\frac{|q|}{2}]![\frac{|q|+1}{2}]!}(z)^{[\frac{m-(-)^{m+i}i}{2}]}(z')^{[\frac{m+(-)^{m+i}i+1}{2}]}}{2^m\sqrt{[\frac{m+i}{2}]![\frac{|q|+m+i}{2}]![\frac{m+1-i}{2}]![\frac{|q|+m+1-i}{2}]!}}. \tag{30}$$

The charged parabose coherent state for $q < 0$ is

$$|q, z, z'\rangle = N_q(z,z')\sum_{m=0}^{\infty}\sum_{i=1}^{m+1}\frac{(z)^{[\frac{m-(-)^{m+i}i}{2}]}(z')^{[\frac{m+(-)^{m+i}i+1}{2}]}}{2^m\sqrt{[\frac{m+i}{2}]![\frac{|q|+m+i}{2}]![\frac{m+1-i}{2}]![\frac{|q|+m+1-i}{2}]!}}|m, |q|+m; i\rangle \tag{31}$$

with the normalization constant for $q < 0$

$$\begin{aligned}(N_q)^{-2} &= \sum_{m=0}^{\infty}\sum_{i=1}^{m+1}\frac{|z|^{2[\frac{m-(-)^{m+i}i}{2}]}|z'|^{2[\frac{m+(-)^{m+i}i+1}{2}]}}{2^{2m}[\frac{m+i}{2}]![\frac{|q|+m+i}{2}]![\frac{m+1-i}{2}]![\frac{|q|+m+1-i}{2}]!}\\ &= \left(i\frac{|z|}{2}\right)^{-[\frac{|q|+1}{2}]}J_{[\frac{|q|+1}{2}]}(i|z|)\left(i\frac{|z'|}{2}\right)^{-[\frac{|q|}{2}]}J_{[\frac{|q|}{2}]}(i|z'|).\end{aligned} \tag{32}$$

The inner product of two non-negatively charged parabose coherent states is $(q, q' \geqslant 0)$

$$\langle q, z, z'|q', w, w'\rangle = \delta_{q,q'}\frac{(\frac{i}{2}w^*z)^{-[\frac{q}{2}]}J_{[\frac{q}{2}]}(iw^*z)(\frac{i}{2}(w')^*z')^{-[\frac{q+1}{2}]}J_{[\frac{q+1}{2}]}(i(w')^*z')}{N_q(z,z')N_{q'}(w,w')} \tag{33}$$

and for $(q, q' < 0)$

$$\langle q, z, z'|q', w, w'\rangle = \delta_{q,q'}\frac{(\frac{i}{2}w^*z)^{-[\frac{|q|+1}{2}]}J_{[\frac{|q|+1}{2}]}(iw^*z)(\frac{i}{2}(w')^*z')^{-[\frac{|q|}{2}]}J_{[\frac{|q|}{2}]}(i(w')^*z')}{N_q(z,z')N_{q'}(w,w')} \tag{34}$$

If $q \geqslant 0$ and $q' < 0$, the inner product vanishes. Therefore, the charged parabose coherent states with different charges are orthogonal, but for the same q-sector, the charged parabose coherent states are not orthogonal. Consequently, for the same q-sector the charged parabose coherent states are linearly dependent and overcomplete.

These charged parabose coherent states satisfy the completeness relation

$$\sum_{q=-\infty}^{\infty}\int\frac{\mathrm{d}^2z\,\mathrm{d}^2z'}{\pi^2}\Phi_q(z,z')|q,z,z'\rangle\langle q,z,z'| = I \tag{35}$$

where $\mathrm{d}^2z = r\,\mathrm{d}r\,\mathrm{d}\theta$, $\mathrm{d}^2z' = r'\,\mathrm{d}r'\,\mathrm{d}\theta'$, and

$$\Phi_q(z,z') = \begin{cases}\frac{1}{4}(-i)^{[\frac{q}{2}]+[\frac{q+1}{2}]}J_{[\frac{q}{2}]}(i|z|)K_{[\frac{q}{2}]}(|z|)J_{[\frac{q+1}{2}]}(i|z'|)K_{[\frac{q+1}{2}]}(|z'|) & \text{for} \quad q \geqslant 0\\ \frac{1}{4}(-i)^{[\frac{|q|}{2}]+[\frac{|q|+1}{2}]}J_{[\frac{|q|+1}{2}]}(i|z|)K_{[\frac{|q|+1}{2}]}(|z|)J_{[\frac{|q|}{2}]}(i|z'|)K_{[\frac{|q|}{2}]}(|z'|) & \text{for} \quad q < 0\end{cases} \tag{36}$$

with $K_n(x) = \frac{\pi}{2}i\exp(\frac{in\pi}{2})(J_n(ix) + iN_n(ix))$ a modified Bessel function. This result follows since by the integration formula [6]

$$\begin{aligned}\int_0^{\infty}\mathrm{d}r\,r^{\mu}K_{\nu}(ar) &= 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)\\ &\quad(\mathrm{Re}\,(\mu\pm\nu) > 0, \mathrm{Re}\,(a) > 0)\end{aligned} \tag{37}$$

we find

$$\sum_{q=-\infty}^{\infty}\int\frac{\mathrm{d}^2z\,\mathrm{d}^2z'}{\pi^2}\Phi_q(z,z')|q,z,z'\rangle\langle q,z,z'| = \sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{m+1}|q+m,m;i\rangle\langle q+m,m;i|$$
$$+\sum_{-q=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{m+1}|m,|q|+m;i\rangle\langle m,|q|+m;i|$$
$$=\sum_{n,m=0}^{\infty}\sum_{i=1}^{\min(n,m)+1}|n,m;i\rangle\langle n,m;i| = I. \tag{38}$$

For non-negative integer q, using (28) we can rewrite (36) as ($z = r\mathrm{e}^{\mathrm{i}\theta}$, $z' = r'\mathrm{e}^{\mathrm{i}\theta'}$)

$$\Phi_q(z,z') = \tfrac{1}{4}(r/2)^{[q/2]}(r'/2)^{[(q+1)/2]}N_q^{-2}K_{[\frac{q}{2}]}(r)K_{[\frac{q+1}{2}]}(r') \tag{39}$$

where N_q is the normalization factor of the charged parabose coherent state (27) for non-negative integer q and is a non-negative quantity as shown in (28); $(r/2)^{[q/2]}$ and $(r'/2)^{[(q+1)/2]}$ are also non-negative. Finally, from the integral representation [6], $K_n(r) = \int_0^\infty \mathrm{d}t\mathrm{e}^{-r\cosh t}\cosh nt$, we see that the function $K_n(r)$ is non-negative (r is a non-negative real parameter). This verifies that the weight function $\Phi_q(z,z')$ is strictly non-negative for all z, z' and non-negative integer q. For negative integer q, one can similarly verify that the weight function non-negative. This concludes the proof.

In summary, in this paper we construct a complete set of basis vectors for the two-mode paraboson system. In order $p = 2$, the basis vectors are orthonormal and we construct the associated conserved-charge parabose coherent states. The latter are orthogonal between different q-sectors and are overcomplete within each q-sector. It is important to generalize these constructions to more than the two-mode system and to $p > 2$.

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